Scale-Free Networks

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CMPT 858

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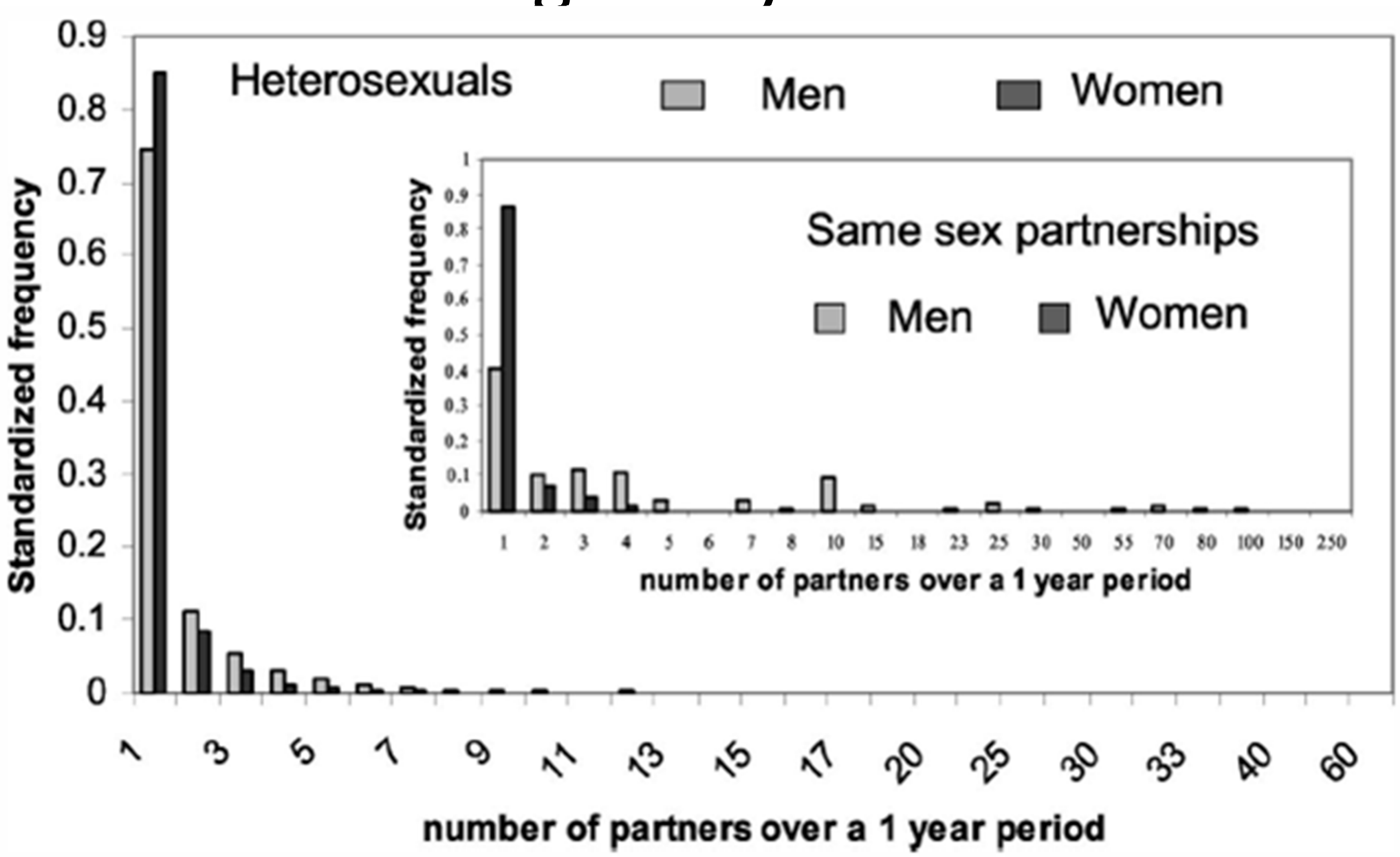
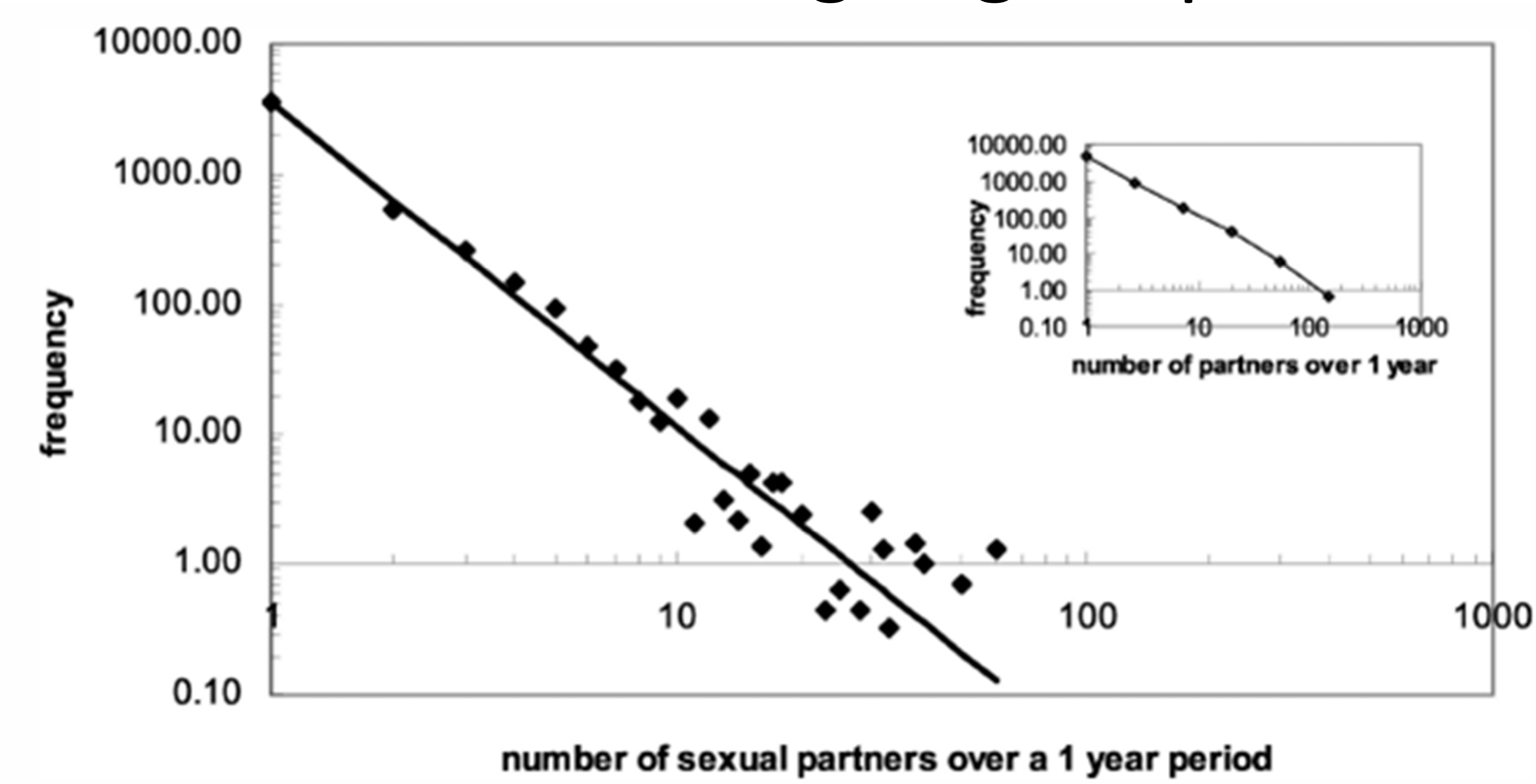
Recall: Heterogeneity in Contact Rates



Schneeberger et al., Scale-Free Networks and Sexually Transmitted Diseases:
A Description of Observed Patterns of Sexual Contacts in Britain and Zimbabwe
, Sexually Transmitted Diseases, June 2004, Volume 31, Issue 6, pp 380-387

Associated Log-Log Graph



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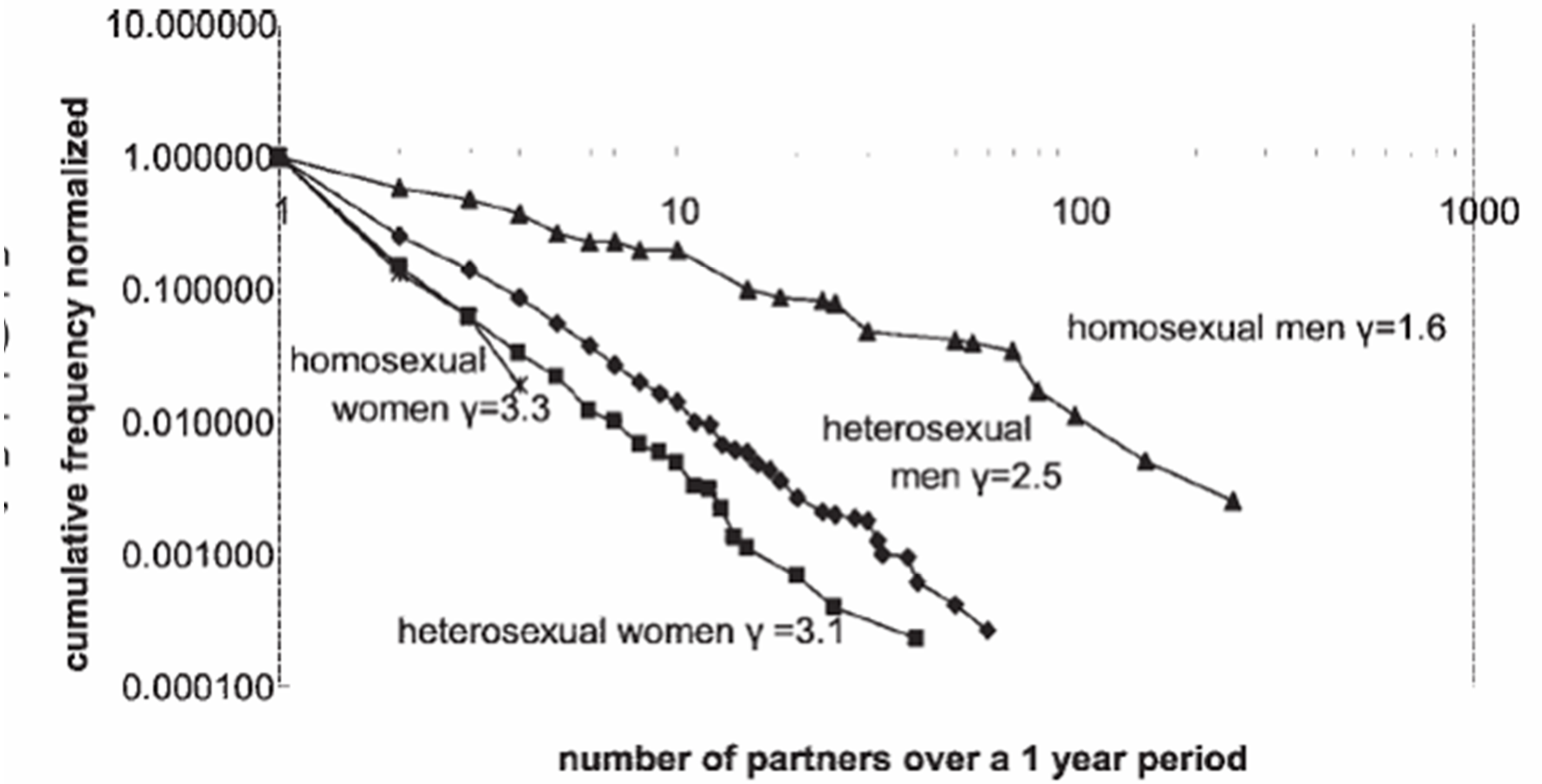
Heterogeneity in Contact Rates

This may significantly affect the spread of infection in the population!



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Intuitive Plausibility of Importance of Heterogeneity

- Someone with high # of partners is both
  - More likely to be infected by a partners
  - More likely to pass on the infection to another person
- Via targeted interventions on high contact people, may be able to achieve great “bang for the buck”
- We may see very different infection rates in high contact-rate individuals
- **How to modify classic equations to account for heterogeneity? How affects infection spread?**

Recall: Classic Infection Term

$$\dot{Y} = c\left(\frac{Y}{N}\right)\beta X - \frac{Y}{D}$$

- Xs are susceptibles, Ys are infectives
- c is contacts per unit time
- β is chance a given contact between an infective and a susceptible will transmit infection

Key Step: Disaggregate by Contact Rate

- We break the population up in to groups according to their rate of contacts
- x_i and y_i are susceptibles, infectives who contact i other people per unit time
  - X is divided into x_0, x_1, …
  - Y is divided into y_0, y_1, …

First Attempt

This rate of contact used to be a single constant (c), but now we've captured the Heterogeneity in rates!

This is the total number of Infected people

$$\dot{y}_i = i\left(\frac{\sum_{j=1}^{\infty} y_j}{N}\right)\beta x_i - \frac{y_i}{D}$$

- Here we are capturing the higher levels of risk for someone of activity class *i* as *i* increases (due to higher contact rates)
- Problem:
  - We are assuming that our *i* contacts are equally spread among other people – in fact, they are skewed towards *others* with a high # of contacts!
  - People with high #s of contacts are more likely to be infected

Revised Formulation

This is the total number of contacts per unit time made by infectives!

$$\dot{y}_i = i\left(\frac{\sum_{j=1}^{\infty} j y_j}{\sum_{j=1}^{\infty} j N_j}\right)\beta x_i - \frac{y_j}{D}$$

This is the total number of *contacts per unit time* made by the entire population.

- x_i and y_i are susceptibles, infectives who contact *i* other people per unit time
- The fraction indicates fraction of *contacts in the population* that are with an infective person
  - *i* times this is the rate of contacts with infectives per unit time experienced by a susceptible in class *i*

Force of Infection

$$\lambda = \beta\left(\frac{\sum_{j=1}^{\infty} j y_j}{\sum_{j=1}^{\infty} j N_j}\right)$$

λ will only grow if y grows!

Reformulating State Equations in λ

$$\lambda = \beta\left(\frac{\sum_{j=1}^{\infty} jy_j}{\sum_{j=1}^{\infty} jN_j}\right)$$

$$\dot{\lambda} \approx \beta\left(\frac{\sum_{j=1}^{\infty} j\dot{y}_j}{\sum_{j=1}^{\infty} jN_j}\right)$$

$$= \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j - \frac{y_j}{D}\right)}{\sum_{j=1}^{\infty} jN_j}\right) = \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j\right) - \sum_{j=1}^{\infty}\frac{jy_j}{D}}{\sum_{j=1}^{\infty} jN_j}\right)$$

$$= \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j\right)}{\sum_{j=1}^{\infty} jN_j} - \frac{\sum_{j=1}^{\infty}\frac{jy_j}{D}}{\sum_{j=1}^{\infty} jN_j}\right) = \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j\right)}{\sum_{j=1}^{\infty} jN_j} - \frac{1}{D}\frac{\sum_{j=1}^{\infty} jy_j}{\sum_{j=1}^{\infty} jN_j}\right) = \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j\right)}{\sum_{j=1}^{\infty} jN_j} - \frac{1}{D}\lambda\right)$$

NDO1

$$\dot{\lambda} \approx \beta\left(\frac{\sum_{j=1}^{\infty} j\left(j\lambda N_j\right)}{\sum_{j=1}^{\infty} jN_j} - \frac{1}{D}\lambda\right) = \beta\left(\frac{\sum_{j=1}^{\infty} j^2\lambda N_j}{\sum_{j=1}^{\infty} jN_j}\right) - \frac{\lambda}{D} = \beta\lambda\left(\frac{\sum_{j=1}^{\infty} j^2 N_j}{\sum_{j=1}^{\infty} jN_j}\right) - \frac{\lambda}{D}$$

$$= \beta\lambda\left(\frac{\sum_{j=1}^{\infty} j^2 N_j}{\sum_{j=1}^{\infty} jN_j}\right)\frac{\left(\frac{1}{N}\right)}{\left(\frac{1}{N}\right)} - \frac{\lambda}{D} = \beta\lambda\frac{\sum_{j=1}^{\infty}\left(j^2\frac{N_j}{N}\right)}{\sum_{j=1}^{\infty}\left(j\frac{N_j}{N}\right)} - \frac{\lambda}{D} = \beta\lambda\frac{E\left[j^2\right]}{E\left[j\right]} - \frac{\lambda}{D}$$

$$= \lambda\left(\beta\frac{E\left[j^2\right]}{E\left[j\right]} - \frac{1}{D}\right)$$

Slide 12

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Reformulated Equation

$$\dot{\lambda} = \lambda\left(\beta\frac{E\left[j^2\right]}{E\left[j\right]} - \frac{1}{D}\right)$$

- This is exactly like the normal SIR system, with X = 1,c= $\frac{E\left[j^2\right]}{E\left[j\right]}$
- R_0 is $$\beta\frac{E\left[j^2\right]}{E\left[j\right]}D$$

Reformulating in More Familiar Terms

$$\sigma^2 = Var(j) = E\left[\left(j - E[j]\right)^2\right] = E\left[j^2\right] - \left(E[j]\right)^2$$

$$c = \frac{E\left[j^2\right]}{E[j]} = \frac{\left(E\left[j^2\right] - E[j]^2\right) + E[j]^2}{E[j]} = \frac{\sigma^2 + m^2}{m} = m + \frac{\sigma^2}{m}$$

$$R_0 = \beta c D = \beta \frac{E\left[j^2\right]}{E[j]} D = \beta\left(m + \frac{\sigma^2}{m}\right) D$$

R_0 rises proportional to the coefficient of variation (ratio of the variance to mean)!

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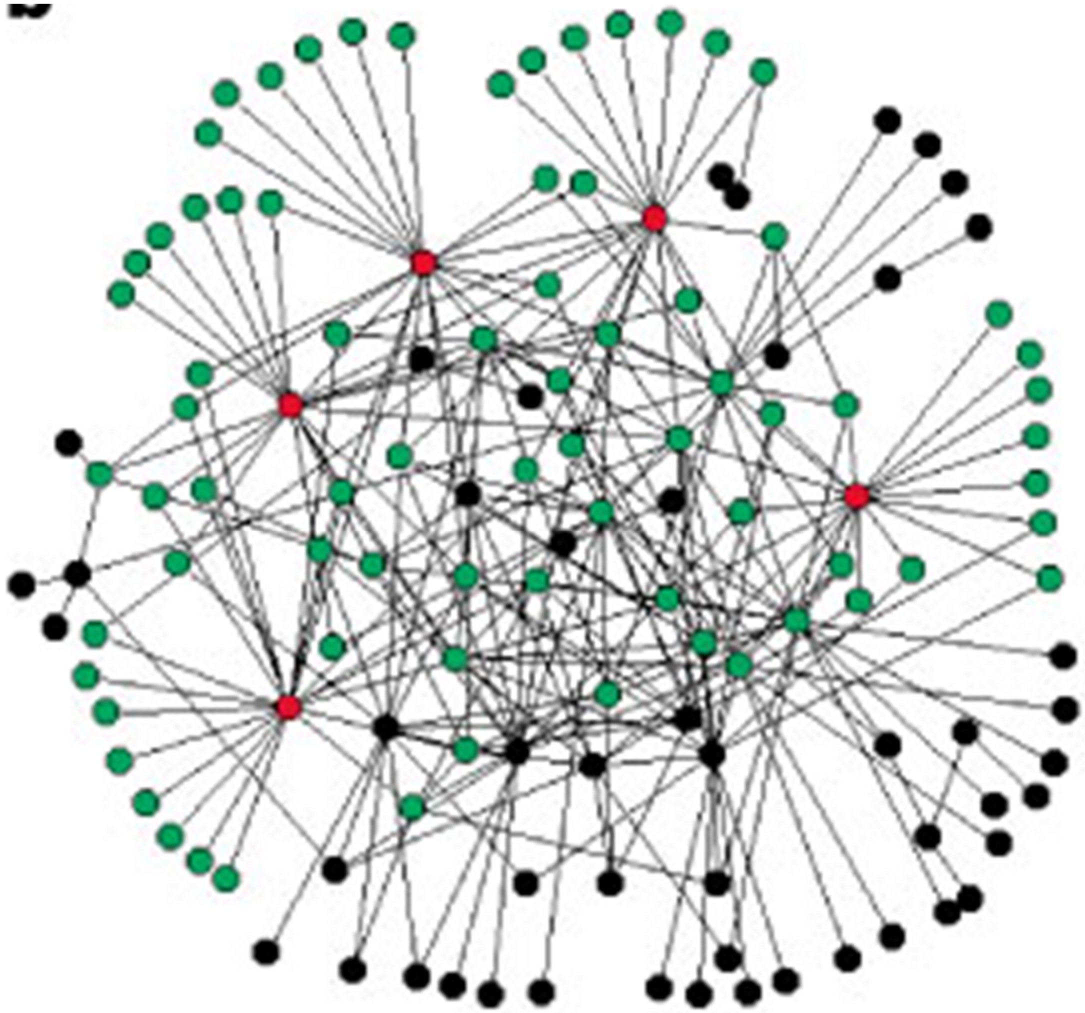

Albert, Jeong and Barabási, Nature 406, 378-382(27 July 2000)